On Vertex Cover with Fractional Fan-Out Bound

Satoshi Fujita
Department of Information Engineering, Hiroshima University
Higashi-Hiroshima, 739-8527, Japan

***Abstract*—In this paper, we consider a new variant of the minimum weight vertex cover problem (MWVC) in which** *each vertex can cover a fractional amount of edges incident on it*. **For example, if the degree of a vertex is five and the designated fraction is $2/3$, then it can cover at most $\lceil (2/3) \times 5 \rceil = 4$ edges among five incident edges. This problem is motivated by a sustainable monitoring of the environment by a set of agents placed at the vertices of graph G so that the failure of agents can be easily recovered by its nearby agents within a short time. This paper investigates the computational complexity of this optimization problem. More specifically, we show that the number of vertices of odd degree, denoted as n_o, plays a key role in determining the hardness of the problem, so that when the given fraction is $1/2$, the complexity of the problem increases as n_o increases, i.e., it can be solved in polynomial time when $n_o = \mathcal{O}(1)$, although it cannot be approximated within an arbitrary constant factor when $n_o = n$, where n is the total number of vertices in the given graph.**

***Index Terms*—Minimum weight vertex cover problem, computational complexity, APX-hardness.**

I. Introduction

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E, and let w be a weight function from V to $\mathbb{R}^+$. In the following, we call $w(u)$ the weight of vertex u. A **vertex cover** of G is a subset of V such that any edge in E has at least one end-vertex in the subset. Minimum weight vertex cover problem (MWVC, for short) is the problem of finding a vertex cover with minimum weight. In the following, we refer to MWVC with $w(u) = 1$ for all u's as MVC, for brevity.

A. Related Work

The computational complexity of MWVC and MVC has been extensively investigated during past decades. MVC is NP-hard even for planar graphs [8], while it is polynomially solvable for bipartite graphs, chordal graphs, graphs with bounded treewidth, and others [3]. It has a simple 2-approximation algorithm based on the maximal matching, while it is hard to approximate within an arbitrary constant factor unless $\mathcal{P} = \mathcal{NP}$, i.e., APX-hard[1] [11].

Several variants of MVC have also been investigated in the literature. Minimum connected vertex cover problem is the problem of finding a minimum vertex cover which induces a single connected component of G. This problem is known to be NP-hard even for planar graphs of maximum degree 4 [7], which was later refined to be planar *bipartite* graphs with maximum degree 4 [5], while it is polynomially solvable when the degree of the input graph is bounded by 3 [12]. As for the approximability of the problem, it is known that the minimum connected vertex cover problem is 2-approximable [2]. Capacitated vertex cover problem (CVC, for short) is a variant of MWVC in which each vertex is given a capacity and the number of incident edges covered by a vertex is bounded by the capacity of the vertex [9]. It is known that CVC is 2-approximable, and several fixed-parameter algorithms have been proposed for CVC and its variants [10]. Maximum partial vertex cover problem is the problem of, given two integers $k \geq 0$ and $t \geq 0$, determining whether there exists a vertex subset $U \subseteq V$ of size at most k such that U covers at least t edges in G [4]. It admits a 2-approximation similar to other variants of MVC [4], [6] and the connection to fixed-parameter algorithms is deeply investigated in [10].

B. Our Contribution

In this paper, we consider a new variant of MWVC in which *each vertex can cover a fractional amount of edges incident on it*. The problem we will consider in this paper is formally stated as follows. Let ρ be a real in $(0, 1]$. A **vertex cover with fan-out bound** ρ is a vertex subset $U(\subseteq V)$ and a function f: $E \to U$ such that:

[1]A problem is said to be APX-hard if there is a PTAS reduction from every problem in APX to that problem, where APX is a subclass of $\mathcal{NP}$ problems which admit constant-factor approximation algorithms. It is known that any APX-hard problem does not admit an approximation scheme with arbitrarily small approximation factor, unless $\mathcal{P} = \mathcal{NP}$.

This paper was supported by the Grant-in-Aid for Scientific Research No.24500082 of the JSPS.

- $f(e) = u$ implies $u \in e$, i.e., every edge must be covered by a vertex in U incident on it, and
- for all $u \in U$, the number of edges assigned to u must not exceed $\lceil \rho \times d(u) \rceil$, where $d(u)$ denotes the degree of u in G.

MWVC with fan-out bound ρ, abbreviated as ρ-MWVC hereafter, is the problem of finding a minimum weight vertex cover under fan-out bound ρ. Note that this is a generalization of MWVC since "$\rho = 1$" corresponds to the ordinary MWVC and this is a special case of CVC so that the capacity of the vertices is controlled by a single parameter ρ.

A motivation of introducing fan-out bound to MWVC is to realize a *sustainable monitoring of the environment* by several agents. In an ordinary setting, given a network modeled by graph G, the agent placed at a vertex is expected to cover all edges incident on it (e.g., hallways in a musium). In addition, to reduce the cost of the monitoring as much as possible, the number of agents to be deployed must be minimized, which is attained by solving MWVC for graph G (although it is NP-hard). In other words, it is commonly requested that agents are placed at the vertices so that *the number of edges covered by two agents is as small as possible* (if every edge is covered by exactly one end-vertex, then it naturally derives a *minimal* vertex cover of G). However, such a rigid assignment is not robust against failures. In fact, if an agent crushes, we need to reconfigure a large portion of the assignment of agents, and in many cases, we need to deploy a new agent as a substitute of the crushed one which generally takes (at least) few hours before completing the recovery. On the other hand, if the fan-out of each vertex u is bounded by $\lceil \rho \times d(u) \rceil$ for an appropriate $\rho < 1$, the role of a crushed agent can be efficiently taken over by its nearby agents, which significantly reduces the recovery time. In addition, by bounding the number of edges actually monitored by each agent by a certain value, we can reduce the load of each agent, while it increases the number of agents necessary to cover all edges. As such, parameter ρ used in the definition of the problem effectively controls the residual availability of each agent, which strongly motivates us to investigate the property of the new problem, including the computational complexity and the existence of polynomial time algorithms for special cases (note that since ρ-MWVC is at least as hard as MWVC, we need to make some restrictions on the problem to derive positive results).

Main results derived in this paper are summarized as follows: 1) $(1/2)$-MWVC is polynomially solvable if the number of vertices of odd degree is bounded by $\mathcal{O}(1)$; and 2) $\left(\frac{r}{2r-1}\right)$-MWVC for $(2r-1)$-regular graphs[2] is APX-hard for any fixed $r \geq 3$. Since

$$\left\lceil \frac{r}{2r-1} \times (2r-1) \right\rceil = \left\lceil \frac{1}{2} \times (2r-1) \right\rceil = r$$

holds, $\left(\frac{r}{2r-1}\right)$-MWVC is equivalent to $(1/2)$-MWVC for $(2r-1)$-regular graphs. Thus the above results indicate that when $\rho = 1/2$, the complexity of ρ-MWVC strongly depends on the number of vertices of odd degree, namely, although the problem is polynomially solvable if the number of such vertices is small, it is hard to approximate within an arbitrary factor if the number of such vertices is n even if the underlying graph is restricted to be regular.

The remainder of this paper is organized as follows. Section II describes elementary results. Section III describes positive results including a polynomial time algorithm for a subclass of instances. Section IV gives a proof of the APX-hardness. Finally, Section V concludes the paper with future work.

II. Elementary Results

Since any subset $U \subseteq V$ can cover at most $\sum_{u \in V} \lceil \rho \times d(u) \rceil$ edges under fan-out bound ρ, parameter ρ should satisfy the following inequality:

$$\sum_{u \in V} \lceil \rho \times d(u) \rceil \;\leq\; |E|. \tag{1}$$

In the following, we assume $\rho \geq 1/2$, without loss of generality. The reader should note that although "to be $\rho \geq 1/2$" is a sufficient condition to have a feasible solution (see Lemma 1 below), this is not necessary in general, since if the given graph is 3-regular, any $\rho > 1/3$ admits a feasible solution.

Lemma 1: For any G, $\rho \geq 1/2$ is a sufficient condition for the existence of a feasible solution to ρ-MWVC.

Proof: It is enough to show that there is a feasible solution when $\rho = 1/2$ and $U = V$. If every vertex in V has even degree, we can calculate a feasible solution in the following manner: 1) identify a cycle C in G and fix the orientation of edges in the cycle so that it forms a directed cycle; 2) for each edge $e = \{u, v\}$ in C, if it is oriented from u to v, then assign e to u; 3) remove all edges in C from G to have a new graph G'. By construction, a feasible solution for G can be obtained from a feasible solution for G' and the assignment given to C (recall that we are assuming $\rho = 1/2$). Hence by repeating similar steps until the resulting graph becomes

[2]Graph G is said to be r-regular (or simply regular) if all vertices have the same number of adjacent vertices.

empty (note that such a recursion always terminates because we are assuming that every vertex in G has even degree), we have a feasible solution for G.

If G contains a vertex of odd degree, on the other hand, we can extend the above argument in the following manner. Let W be the set of vertices with odd degree. Note that $|W|$ must be even since $\sum_{u \in V} d(u) = 2|E|$. By connecting $|W|/2$ arbitrary pairs of the vertices in W by dummy edges, we have a super-graph G'' of G such that all vertices have even degree. Thus, after constructing a vertex cover of G'' with $\rho = 1/2$ using the above argument, we can obtain a solution for G by simply omitting dummy edges in G''. Hence the lemma follows. ∎

Corollary 1: If every vertex in G has even degree, $(1/2)$-MWVC of G can be calculated in linear time.

With the above notions, we can derive a naive approximation scheme for solving ρ-MVC as follows. Let Δ and Δ^* denote the maximum and average degree of graph G, respectively. By assumption, each vertex in G can cover at most $\lceil \rho\Delta \rceil \leq \rho\Delta + 1$ incident edges. Thus, to cover all of $|E|$ edges, any feasible solution must contain at least

$$\frac{|E|}{\rho\Delta + 1}$$

vertices. As Lemma 1 claims, when $\rho \geq 1/2$, $U = V$ is a feasible solution to the problem. Since $|V|$ can be represented as $|V| = 2|E|/\Delta^*$, the approximation ratio of such a naive scheme is at most

$$\frac{|V|}{|E|/(\rho\Delta + 1)} = \frac{2(\rho\Delta + 1)}{\Delta^*}.$$

Thus we have the following proposition.

Proposition 1: For any $\rho \geq 1/2$, there is an approximation scheme for solving ρ-MVC with approximation ratio

$$\frac{2(\rho\Delta + 1)}{\Delta^*}$$

where Δ and Δ^* are the maximum and average degree of G, respectively.

Corollary 2: If the given G is r-regular, the approximation ratio of the above naive scheme is $2\rho + 2/r$.

Thus for example, when $r = 5$ and $\rho = 2/3$, the approximation ratio of the scheme is calculated as $2 \times 2/3 + 2/5 \leq 1.734$.

III. Positive Results

As Lemma 1 claims, if $\rho = 1/2$ and G contains no vertex of odd degree, then $U = V$ is the unique solution to the problem (although there might exist exponential number of candidates for function f). However, if G contains vertices of odd degree, we could "reduce" the size of U from V by carefully assigning edges to the vertices. This section first describes a heuristic scheme to calculate such a solution. The reader should note that although we could bound the approximation ratio of the resulting scheme by a certain value as in Proposition 1, as will be proved in the next section, the approximation ratio could not be arbitrarily small, unless $\mathcal{P} = \mathcal{NP}$.

Let ϕ denote a $(1/2)$-MWVC with $U = V$ obtained by applying the procedure described in the proof of Lemma 1. In the following, to make the exposition clear, we identify function ϕ with the orientation of edges satisfying the constraint (see the proof of Lemma 1 as for the meaning of the orientation of edges). Given such an initial configuration, we can reduce the weight of U in the following steps. Recall that by construction, under the assignment ϕ, each vertex u has at least $\lfloor d(u)/2 \rfloor$ "incoming" edges and at most $\lceil d(u)/2 \rceil$ "outgoing" edges. Let $a(u)$ be a variable representing the number of incident edges of u which can be changed from an incoming edge to an outgoing edge without violating the constraint on the fan-out bound ρ for u. Since u can have at most $\lceil \rho \times d(u) \rceil$ outgoing edges, under ϕ, $a(u)$ is initialized to $\lceil \rho \times d(u) \rceil - \lceil d(u)/2 \rceil$ or $\lceil \rho \times d(u) \rceil - \lfloor d(u)/2 \rfloor$ for each u (note that the above value is not negative since $\rho \geq 1/2$).

Let u^* be a vertex in U. Vertex u^* can be "removed" from U by changing all of its outgoing edges to incoming edges. More specifically, an outgoing edge of u^*, say e^*, can be changed to be an incoming edge by conducting the following steps:

1) identify a directed path starting from e^* which ends up with a vertex v with $a(v) > 0$,
2) change the direction of the path to be "from v to u^*," and
3) decrement $a(v)$ by one and increment $a(u^*)$ by one.

Note that such a change of the direction of a path does not change the value of $a(\cdot)$ of non-terminal vertices on the path; i.e., it works as an alternating path used in max-flow algorithms.

A possible heuristic to find a small U is to repeat the above "path reversal" operation until no additional removal can be applied. Unfortunately, such a reversible path does not always exist in G even if there remains a vertex $v \in U$ with $a(v) > 0$; i.e., the goodness of the solution depends on the order of removals of the vertices from U. In fact, there is an instance such that we can remove three vertices from U by removing vertices in an appropriate order, but by using a wrong order of removals, we can not remove more than two vertices (see Figure 1 for illustration).

As will be shown in the next section, the problem

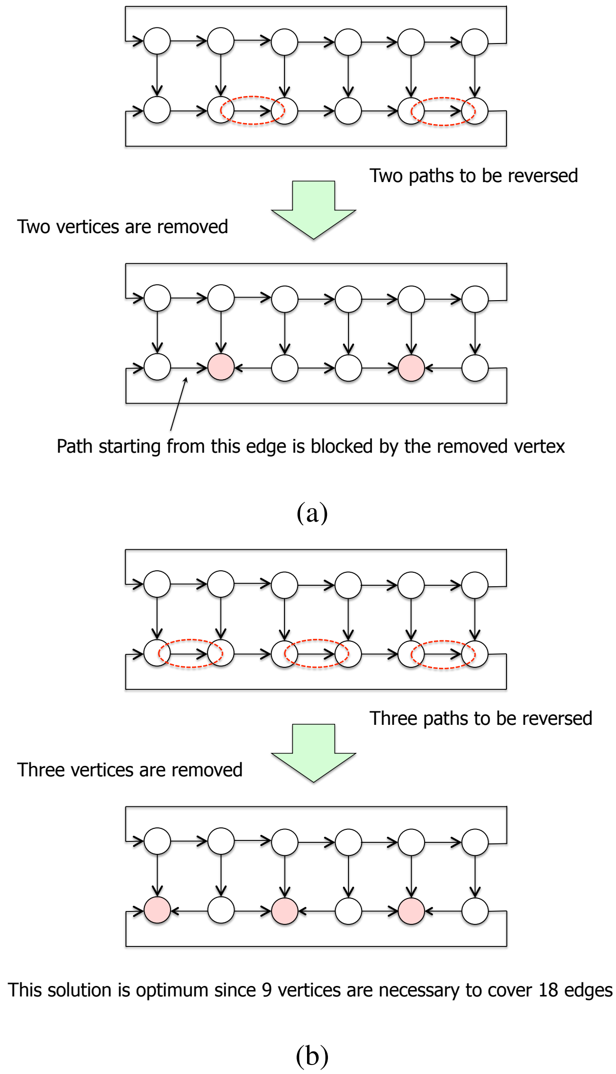


Fig. 1. Counter-example for the naive path reversal scheme.

of finding a solution with minimum weight is in fact NP-hard and difficult to approximate within an arbitrary approximation ratio. However, by setting an appropriate constraint on the set of instances, we can calculate an optimum solution in polynomial time as the following theorem claims.

Theorem 1: If the number of odd vertices in G is $\mathcal{O}(1)$, then $(1/2)$-MWVC can be solved in polynomial time.

Proof: Let $W(\subseteq V)$ denote the subset of vertices of odd degree. Since any vertex of even degree does not have more outgoing edges than incoming edges under any configuration for $(1/2)$-MWVC, the total number of paths contributing to the removal of vertices from $U = V$ is at most $|W|$. In other words, if we assume that the set of vertices contributing to the removal of vertices is $Y(\subseteq W)$ and G has a sufficiently large connectivity to

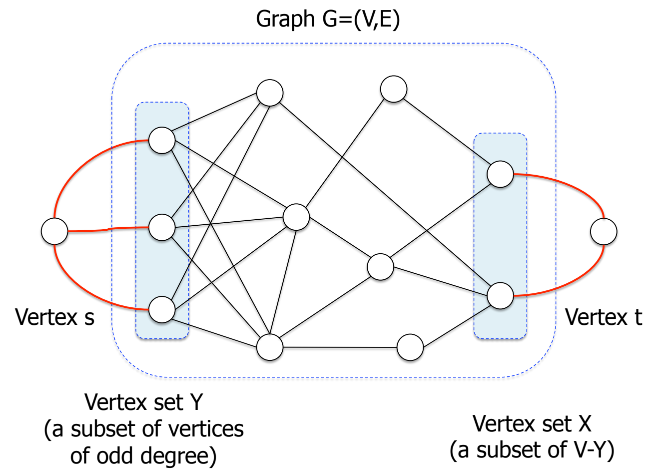


Fig. 2. Graph G' in the proof of theorem 1 (the capacity of red edges is infinity and the capacity of the other edges is one).

allow edge-disjoint paths connecting Y and the set of removed vertices, say X, in G so that all edges incident on X are "fully" used, then we can maximize the weight of removed vertices by solving a Knapsack problem in which:

1) each item corresponding to a vertex in $V - Y$ is assigned cost (i.e., $d(\cdot)$ in our terminology) and value ($w(\cdot)$ in our terminology), and
2) the total value of selected items is maximized subject to the total cost of selected items is bounded by $|Y|$.

Thus by examining the feasibility of each subset of $V-Y$ as a set of removed vertices and by taking the maximum of the resulting value over all feasible subsets, we have a maximum, feasible subset of vertices which can be removed from G with the aid of subset Y, where the feasibility check for subset $X(\subseteq V - Y)$ proceeds as follows:

- Attach vertices s and t to G so that s connects to all vertices in Y by links of infinite capacity and t connects to all vertices in X by links of infinite capacity (see Figure 2 for illustration);
- Calculate the maximum flow from s to t by assuming that each edge in G has unit capacity; and
- If the size of the maximum flow is smaller than $\sum_{u \in X} d(u)$, then X is not feasible, otherwise, it is feasible, i.e., there exists a set of edge-disjoint paths from Y to X so that all edges incident on X are incoming edges.

The number of subsets of $V - Y$ to be examined in the above procedure is polynomial since the size of each subset is assumed to be constant (i.e., there are at most $n^{\mathcal{O}(1)}$ such subsets), and for each subset, we can calculate the feasibility of the subset in polynomial time by using max-flow algorithm. Thus an optimum solution with respect to $Y(\subseteq W)$ is calculated in polynomial time. In addition, since the number of subsets of W is polynomial, the total running time of the overall scheme is polynomial. Hence the theorem follows. ∎

IV. APX-Hardness

This section proves the APX-hardness of ρ-MWVC. More precisely, we prove the following theorem.

Theorem 2: For any $\rho = \frac{r}{2r-1}$ with fixed integer $r \geq 3$, ρ-MWVC for $(2r-1)$-regular graphs is APX-hard.

The proof of the theorem is based on an L-reduction from MVC for cubic graphs which is known to be APX-complete [11]. Given two NP optimization problems $\mathcal{F}$ and $\mathcal{G}$ and a polynomial time transformation ξ from instances of $\mathcal{F}$ to instances of $\mathcal{G}$, we say that ξ is an L**-reduction** from $\mathcal{F}$ to $\mathcal{G}$ if there are positive constants α and β such that the following two conditions hold for every instance x of $\mathcal{F}$ [11].

1) Optimum solution of $\xi(x)$ with respect to problem $\mathcal{G}$, denoted by $opt_{\mathcal{G}}(\xi(x))$, is at most α times of the optimum solution of x with respect to problem $\mathcal{F}$, denoted by $opt_{\mathcal{F}}(x)$.
2) For every feasible solution y of $\xi(x)$ with objective value c_2, we can in polynomial time find a solution y' of x with objective value c_1 such that $|opt_{\mathcal{F}}(x) - c_1| \leq \beta|opt_{\mathcal{G}}(\xi(x)) - c_2|$.

It is known that if $\mathcal{F}$ is APX-hard and there is an L-reduction from $\mathcal{F}$ to $\mathcal{G}$, then $\mathcal{G}$ is also APX-hard [11]. Our proof consists of two steps. The first step is a reduction from MVC for cubic graphs to MVC for r-regular graphs (Section IV-A), and the second step is a reduction from MVS for r-regular graphs to $(\frac{r}{2r-1})$-MWVC for $(2r-1)$-regular graphs (Section IV-B).

A. First Step

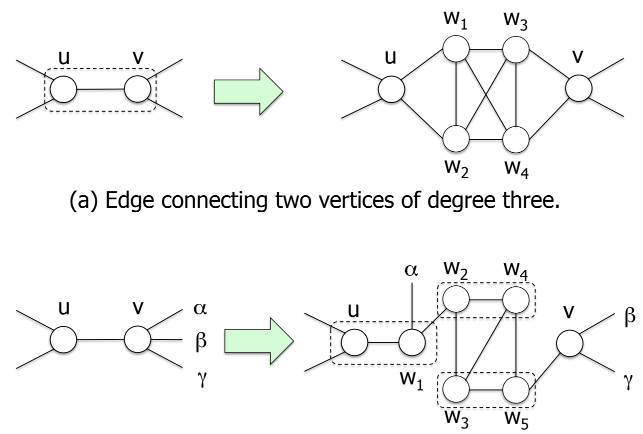
This subsection proves the following theorem.

Theorem 3: MVC for r-regular graphs is APX-hard for any fixed $r \geq 3$.

This theorem is an immediate consequence of the following two lemmas.

Lemma 2: MVC for r-regular graphs is APX-hard, for $r = 3$ and 4.

Proof: Since APX-hardness for $r = 3$ is proved in [1], it is enough to prove the claim for $r = 4$. Let $G = (V, E)$ be a cubic graph, where we assume G is connected, without loss of generality. Before proceeding to the detailed description of the proof, we introduce two notions which play a key role in the proposed reduction. A **perfect matching** of an n-set S is a set of $\lfloor n/2 \rfloor$ disjoint 2-sets drawn from S. Given two edge-disjoint paths p_1 and p_2 sharing a vertex in cubic graph G, we say that p_1 **dominates** p_2 if: 1) p_1 passes through a terminal vertex of p_2 and 2) p_2 does not pass through a terminal vertex of p_1. Note that any two edge-disjoint paths p_1 and p_2 which share two vertices in G but are not sharing their end vertices can be transformed into two edge-disjoint paths p'_1 and p'_2 such that p'_1 dominates p'_2, while keeping the set of edges used in these paths.



(a) Edge connecting two vertices of degree three.

(b) Edge connecting vertices of degrees three and four.

Fig. 3. Two transformations used in Lemma 2.

At first, we calculate a perfect matching M of V satisfying the following conditions, which will be referred to as Condition DOM hereafter:

1) pairs in M are connected by a set P of edge-disjoint paths in G; namely, so that every edge in G is used at most once in P;
2) if two paths p_1 and p_2 in P share a vertex, then either p_1 dominates p_2 or p_2 dominates p_1; and
3) the domination relation between paths in P is a partial order on P.

Since $|V|$ is even, V has a perfect matching of cardinality $|V|/2$. In addition, we can find a perfect matching satisfying Condition DOM in polynomial time by repeating local modification starting from arbitrary perfect matching of V.

For each $\{u, v\} \in M$, let $p(u, v) \in P$ denote the path connecting u and v. If $p(u, v)$ consists of one edge, we can increase the degree of u and v by one, by applying the transformation shown in Figure 3 (a). For the other pairs of vertices, we conduct the following operation in an order such that *the processing for a path* $p(u, v)$ *can start only after all paths dominated by* p *have been processed*. The reader should note that under such an ordering, we have a situation such that for each path currently being processed, all vertices on the path, except

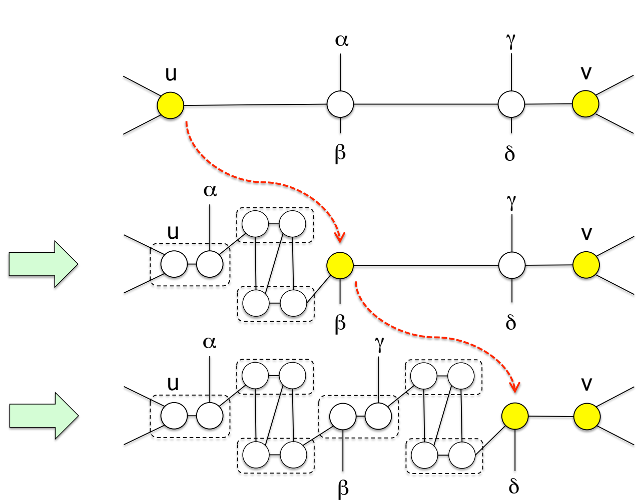


Fig. 4. The movement of a vertex of degree 3 from the position of vertex u to the position of the adjacent vertex of v.

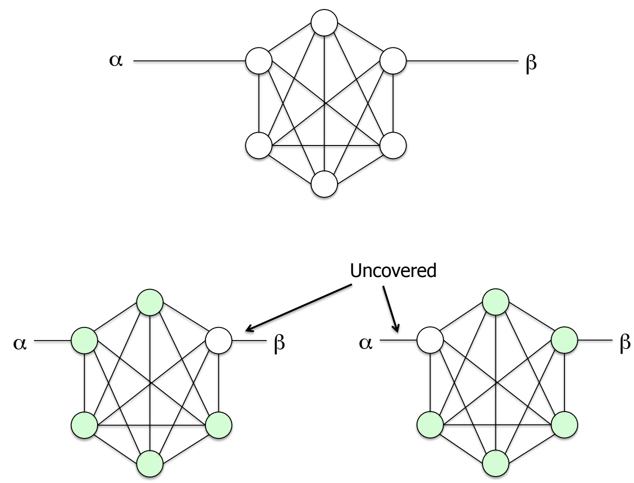


Fig. 5. Component used in the proof of Lemma 3 for $r = 5$ (two figures at the bottom represents vertex covers of the component of cardinality five).

for the terminal vertices, have degree 4. The idea of the transformation is to sequentially "move" the position of a *vertex of degree 3* from the position of u to the adjacent vertex of v on the path, by repeating the transformation Tb represented in Figure 3 (b), where pairs of vertices of degree 3 enclosed by a dashed rectangle in the figure are replaced by a component consisting of vertices of degree 4, by applying the transformation Ta shown in Figure 3 (a). After completing the movement of a vertex of degree 3 to the adjacent vertex of v, we increase the degree of those vertices by applying Ta to them. Figure 4 illustrates the movement in two hops. As such, we can always have a 4-regular graph G' from cubic graph G by applying two operations shown in Figure 3.

The fact that the above transformation from G to G' is in fact an L-reduction is verified as follows. Every application of transformation Ta increases the size of the solution by 3. Every application of transformation Tb increases the size of the solution by 3 and for each pair of the vertices enclosed by a dashed rectangle, an application of Ta increases the size of the solution by 3. Thus, the total amount of increase due to the move of a degree-3 vertex to its neighbor is $3 + 3 \times 3 = 12$. Since the set of paths P is established in an edge-disjoint manner, such a move of a degree-3 vertex occurs at most $|E| \leq 2n$ times before completing the overall transformation. On the other hand, since the maximum degree of G is 3, the size of the optimum solution for G is at least $|E|/3 \geq n/3$. Thus the optimum solution for G' is at most constant times of the optimum solution for G, i.e., the first condition of the L-reducibility holds, and simultaneously, for any vertex cover of G', we can construct a vertex cover of G satisfying the second condition. Thus the claim follows. ■

Lemma 3: MVC for r-regular graphs is APX-hard for any fixed $r \geq 5$.

Proof: Let G' be a 4-regular graph which is obtained from cubic graph G by applying the transformation described in the proof of Lemma 2. We prove the claim by providing a transformation from G' to an r-regular graph for $r \geq 5$, which is a part of the transformation of a given cubic graph G to an r-regular graph G'' (the reader should note that G' is *not* an arbitrary 4-regular graph).

At first, we show that graph G' has a perfect matching in the graph theoretical sense. Recall that the role of Tb is to transform a vertex of degree 4 into six vertices of degree 3 each, and the actual increment of the degree is attained by applying Ta to a part of the component enclosed by a dashed rectangle or two neighboring vertices of degree 3 each. In addition, for each edge in G, the transformation of the edge is conducted at most once. Hence by (imaginarily) suspending the application of Ta to the graph obtained by applying Tb to G (graph shown in Figure 3 (b) represents such a graph), we have a cubic graph to have a perfect matching consisting of edges to which Ta will be applied. Thus, by selecting edges in each copy of the component used in Ta so that $\{\{u, w_1\}, \{w_2, w_3\}, \{w_4, v\}\}$ (see Figure 3 (a) for the notation), we have a perfect matching of the resulting graph G'.

Let $\hat{E}$ be a perfect matching of 4-regular graph G'. Since the number of vertices in G' is even, $\hat{E}$ is an edge cover of G', i.e., each vertex in G' is incident on exactly one edge in $\hat{E}$. Consider the following construction of

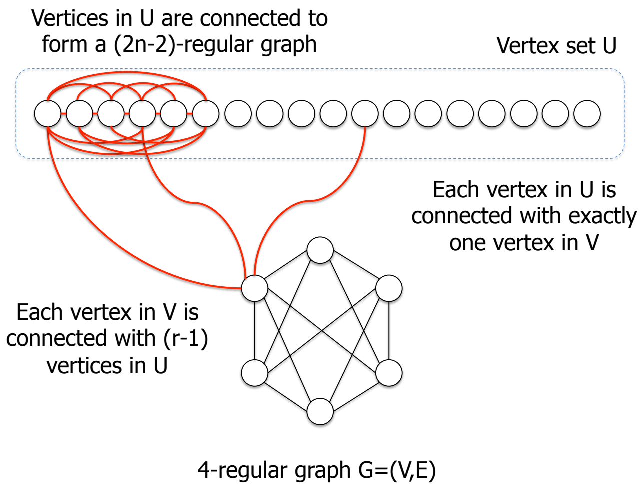


Fig. 6. Transformation used in Lemma 4 ($r = 4$).

an r-regular graph ($r \geq 5$) G'' from G':

1) Prepare $(r-3)|\hat{E}|$ copies of the component depicted in Figure 5, which is obtained by "cutting" an edge in K_{r+1} at the middle point. The reader should note that to cover all edges in the component except for one "open" edge, we must use at least r vertices and there is a subset of r vertices which cover all edges in the component except for one open edge (see two graphs in the bottom of the figure, where green vertices form a vertex cover).
2) Replace each edge $\{u, v\} \in \hat{E}$ by $r-3$ copies of the component each of which connects u and v.

By construction, the resulting graph is r-regular. In addition, if G' has a vertex cover of size c, then G'' has a vertex cover of size $c + r(r-3)|\hat{E}|$, and vise versa. Since $c \geq n/2$ and $|\hat{E}| = n/2$, by Lemma 2, it gives an L-reduction from MVC for cubic graphs to MVC for r-regular graphs. Hence the lemma follows. ■

B. Step Two

Next, we prove the following lemma, which completes the proof of Theorem 2.

Lemma 4: For any $r \geq 3$, there is an L-reduction from MVC for r-regular graphs to $(\frac{r}{2r-1})$-MWVC for $(2r-1)$-regular graphs.

Proof: Let $G = (V, E)$ be an r-regular graph consisting of n vertices. Let U be a set of $(r-1)n$ vertices such that $U \cap V = \emptyset$. We construct a $(2r-1)$-regular graph G' with vertex set $U \cup V$ by connecting those vertices in the following manner (see Figure 6 for illustration):

- Vertices in V are connected as in G, and each vertex in V is connected with $r-1$ vertices in U, so that the degree of those vertices becomes $2r-1$.
- Each vertex in U is connected with exactly one vertex in V and $2r-2$ other vertices in U so that the degree of those vertices becomes $2r-1$, in such a way that the subgraph of G' induced by U is a $(2r-2)$-regular graph. Note that such a connection is always possible for any $n \geq 3$ since it is known that there exists a k-regular graph of order ℓ iff $\ell \geq k+1$ and $k\ell$ is even.

In addition, we assign the "weight" to each vertex in G' in the following manner:

- $w(u) := 1$ for each $u \in V$; and
- $w(v) := \epsilon$ for each $v \in U$, where ϵ is a constant satisfying $\epsilon < \frac{1}{r(n-1)}$

By construction, the total weight of the vertices in U is smaller than the weight of a single vertex in V. Thus for any feasible solution f for G' (with respect to (1/2)-MWVC), we can have a solution f' such that: all vertices in U are selected and the difference to the optimum solution f^* increases by at most one. Thus without loss of generality, we may assume that all vertices in U are selected in every feasible solution for G'.

Given a minimum vertex cover of G of size c, we can construct a solution of $(\frac{r}{2r-1})$-MWVC for G' of size $c+1$ by including all vertices in U into the subset and by determining the assignment of edges incident on U such that: 1) each vertex has $r-1$ incoming edges and $r-1$ outgoing edge connecting to the vertices in U and 2) it has an outgoing edge connecting to a vertex in V. Note that it realizes a weighted vertex cover of G' under fractional bound $\rho = \frac{r}{2r-1}$, since each vertex in V has $r-1$ incoming edges connecting to vertices in U and at most r outgoing edges connecting to vertices in V. Thus the first condition of the L-reducibility holds. Since such a correspondence holds even for optimum solution for the problems, the second condition also holds. Thus the lemma follows. ■

V. Concluding Remarks

This paper proposes a new variant of the minimum weighted vertex cover problem with a fractional fan-out bound. It is clarified that the computational complexity of the problem strongly depends on the number of vertices of odd degree when the designated fraction is 1/2.

Future works are listed as follows:

- To extend Theorem 1 so that polynomially solvable cases will be expanded, e.g., when the number of vertices of odd degree is $\mathcal{O}(\log n)$ or $\mathcal{O}(\log \log n)$. To this end, we need to apply the dynamic programming to the algorithm, while it seems to be complicated since we should examine the feasibility of subset X with respect to the existence of edge-disjoint paths from Y unlike ordinary calculations which merely consider the cost bound.

- To expend Theorem 2 so that a similar claim holds for ρ-MVC. To this end, we need to develop a new technique for the reduction which does not rely on the weight of the vertices.
- To evaluate the sustainability of monitoring systems constructed based on the notion of ρ-MWVC with respect to several metrics such as the fault-tolerance and the convergence speed.

References

[1] P. Alimonti and V. Kann. "Some APX-completeness results for cubic graphs." *Theoretical Computer Science*, 237:123–134, 2000.

[2] E. M. Arkin, M. M. Halldórsson, R. Hassin. "Approximating the tree and tour covers of a graph." *Information Processing Letters*, 47(6):275–282, 1993.

[3] A. Brandstadt, V. B. Le, and J. Spinrad. *Graph Classes: A Survey.* Society for Industrial and Applied Mathematic, Philadelphia, 1999.

[4] N. H. Bshouty and L. Burroughs. "Massaging a linear programming solution to give a 2-approximation for a generalization of the Vertex Cover problem." In *Proc. 15th Symposium on Theoretical Aspects of Computer Science (STACS)*, Lecture Notes in Computer Science, volume 1373, 1998, pages 298–308.

[5] H. Fernau and D. Manlove. "Vertex and Edge Covers with Clustering Properties: Complexity and Algorithms." *Journal of Discrete Algorithms*, 7(2):149–167, 2009.

[6] R. Gandhi, S. Khuller, and A. Srinivasan. "Approximation algorithms for partial covering problems." In *Proc. 28th International Colloquium on Automata, Languages, and Programming (ICALP)*, Lecture Notes in Computer Science, volume 2076, 2001, pages 225–236.

[7] M. R. Garey and D. S. Johnson. "The Rectilinear Steiner Tree Problem is NP-Complete." *SIAM Journal of Applied Mathematics*, 32(4):826–834, 1977.

[8] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman and Company, San Francisco, 1979.

[9] S. Guha, R. Hassin, S. Khuller and E. Or. "Capacitated vertex covering." *Journal of Algorithms,* 48:257–270, 2003.

[10] J. Guo, R. Niedermeier, S. Wernicke. "Parameterized Complexity of Generalized Vertex Cover Problems." In *Algorithms and Data Structures*, Lecture Notes in Computer Science, volume 3608, 2005, pages 36-48.

[11] C. H. Papadimitriou and M. Yannakakis. "Optimization, approximation, and complexity classes." *J. Comput. Systems Sci.*, 43:425–440, 1991.

[12] S. Ueno, Y. Kajitani, and S. Gotoh. "On the non separating independent set problem and feedback set problem for graphs with no vertex degree exceeding three." *Discrete Mathematics*, 72(1-3):355–360, 1988.