Solvability for The Maximum Legal Firing Sequence Problem of Conflict-Free Petri Nets with Inhibitor Arcs

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***Abstract*— The subject of this paper is the maximum legal firing sequence problem (MAX-INLFS) for inhibitor-arc Petri nets IN. It is well-known that modeling capability of inhibitor-arc Petri nets is equivalent to that of Turing machines, and MAX-INLFS has wide applications to fundamental problems of Petri net such as the marking reachability problem, the scheduling problem, and so on. It is known that, when IN has weighted forward conflict-free structure and has only one place (called a rivet) to which at least one inhibitor-arc is incident, MAX-INLFS can be solved in pseudo-polynomial time if weights of all edges entering the rivet are equivalent; otherwise it is NP-hard. In this paper, when IN has more than one rivet rv, we show that MAX-INLFS can be solved in in $O(2^{|RV|}|P||X|)$ time, where RV is a set of rivets in IN.**

I. Introduction

An *inhibitor arc* (or simply an *inhibitor*) is a special directed edge (p, t) of unit weight, from a place p to a transition t such that, whenever p has a token, t cannot be fired. Such a place p is called a *rivet*. An inhibitor-arc Petri net $IN = (P, T, I, E, \alpha, \beta)$ consists of a Petri net (called the *underlying* Petri net) with any set of inhibitor arcs added. In figures of this paper, any inhibitor arc is represented as a dashed line terminating with a small circle attached to a transition. It is shown in [1] (see also [2]) that modeling capability of inhibitor-arc Petri nets is equivalent to that of Turing machines since inhibitor-arc Petri nets can test "zero" (that is, whether a place has at least one token or not).

The Legal Firing Sequence problem **INLFS** of inhibitor-arc Petri nets is defined by "Given an inhibitor-arc Petri net IN, an initial marking M_0 and a firing count vector X, find a firing sequence, or a sequence of transitions, which is legal on M_0 with respect to X." A component $X(t)$ of X denotes the prescribed total firing number of a given transition t. Without loss of generality we assume $X(t) > 0$ for any $t \in T$. We say that a firing sequence δ is *legal* on an initial marking M_0 if and only if the first transition of the sequence is can be fired at M_0 and the rest can be fired one after another subsequently. If such δ satisfies that each transition t appears exactly $X(t)$ times in δ then we say that δ is legal on M_0 with respect to X.

Let us introduce the Maximum Legal Firing Sequence problem **MAX-INLFS** defined as follows (see Fig. 1): "Given an inhibitor-arc Petri net IN, an initial marking M_0 and a firing count vector X, find a firing sequence δ such that δ is legal on M_0 within X: (i) δ is legal on M_0 and $\overline{\delta} \leq X$ (meaning that $\overline{\delta}(t) \leq X(t)$ for any $t \in T$); (ii) the length $|\overline{\delta}|$ of δ is maximum among those sequences satisfying (i), where $\overline{\delta}(t)$ is the total number of occurrences of t in δ for any $t \in T$." Let **LFS** or **MAX-LFS**, respectively, denote **INLFS** or **MAX-INLFS** for the underlying Petri net N of IN (that is, all inhibitor arcs of IN are removed). **MAX-INLFS** has wide applications to fundamental problems of Petri net such as the marking reachability problem, the scheduling problem, and so on.

There are many related results for **LFS**, **MAX-LFS**, **INLFS** and **MAX-INLFS**. It is shown in [3] that **INLFS** can be solved in $O(|X|)$ time for any inhibitor-arc Petri net with unweighted state machine structure (that is, the underlying Petri net is an unweighted state machine) if IN has only one rivet and is *non-adjacent type* (see [3] for the definition). On the other hand, **RINLFS** (a decision problem of **INLFS**) is NP-hard even if the following condition (1) or (2) holds: (1) IN has unweighted state machine structure and has at least three rivets, or (2) IN has unweighted forward conflict-free structure and $X(t) = 1$ for any $t \in T$. Note that NP-hardness under the above condition (1) or (2) is proved when the number of rivets in IN is not constant. It is shown in [4] that **MAX-LFS** for a weighted conflict-free Petri net can be solved in $O(|E||X|)$. Furthermore **MAX-INLFS** can be solved in $O(|P||X|)$ time when IN has weighted marked graph structure (that is, the underlying Petri net is a weighted marked graph) and has only one rivet. It is shown in [5] that, when IN has weighted forward conflict-free structure (that is, the underlying Petri net is a weighted forward conflict-free) and has only one rivet rv, (1) **MAX-INLFS** can be solved in $O(|P||X|)$ time if weights of all edges $(t, rv) \in E$ are equivalent; (2) otherwise **RINLFS** is NP-hard.

In this paper, when IN has weighted forward conflict-free structure (that is, the underlying Petri net is a weighted forward conflict-free) and has more than one rivet rv, **MAX-INLFS** can be solved in $O(2^{|RV|}|P||X|)$ time, where RV is a set of rivets in IN.

II. Preliminaries

A *Petri net* is a bipartite digraph $N = (P, T, E, \alpha, \beta)$, where P is the set of *places*, T is that of *transitions* such that $P \cap T = \emptyset$, and $E = E_{pt} \cup E_{tp}$ is an edge set such that E_{pt} consists of edges from P to T with weight function $\alpha : E_{pt} \to Z^+$ (non-negative integers) and E_{tp} consists of edges from T to P with weight function $\beta : E_{tp} \to Z^+$, In all figures in this paper, edge weight one is not shown for simplicity.

We denote an inhibitor arc from $u \in P$ to $v \in T$ as $(u, v)_i$. Petri nets with inhibitor arcs are referred to as *inhibitor-arc Petri nets*, denoted as $IN = (P, T, I, E, \alpha, \beta)$, We used the notation N for an ordinary Petri net (without inhibitor arcs) and IN for an inhibitor-arc Petri net unless otherwise stated. Let ${}^\bullet v = \{u \in P \cup T \mid (u, v) \in E\}$ and $v^\bullet = \{u' \in P \cup T \mid (v, u') \in E\}$. Note that inhibitor arcs are ignored in these definitions. Let ${}^\circ v = \{u \in P \mid (u, v)_i \in I\}$ and $v^\circ = \{u' \in T \mid (v, u')_i \in I\}$.

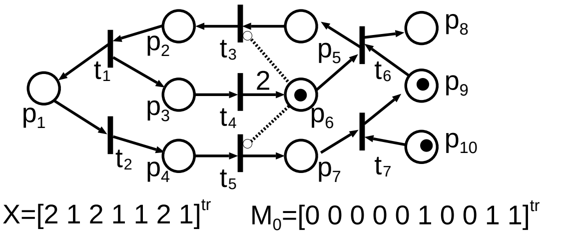

Fig. 1. An example: an inhibitor-arc Petri net IN for which an optimum solution of **MAX-INLFS** is $\delta = t_6t_3t_1t_2t_5t_7t_4t_6$ with $\overline{\delta} = [1,1,1,1,1,2,1] \le X$.

We denote $RV = \{p \in P \mid p^{\circ} \neq \emptyset\}$. Let $T_s = {}^{\bullet}RV$ and $P' = \{rv \in RV \mid M(rv) > 0\} \neq \emptyset$.

A *marking* M for N is a function $M : P \to Z^{+}$, and $|M|$ denotes the total sum of $M(p)$ over all $p \in P$. A transition t of Petri net N is *enabled* at a marking M of N (denoted as $M[t\rangle$) if $M(p) \ge \alpha(p,t)$ for any $p \in {}^{\bullet}t$. *Firing* such t on M is to define a marking M' such that, for any $p \in P$, we have $M'(p) = M(p) + \beta(t,p)$ if $p \in t^{\bullet} - {}^{\bullet}t$, $M'(p) = M(p) - \alpha(p,t)$ if $p \in {}^{\bullet}t - t^{\bullet}$, $M'(p) = M(p) - \alpha(p,t) + \beta(t,p)$ if $p \in {}^{\bullet}t \cap t^{\bullet}$ and $M'(p) = M(p)$ otherwise. We denote as $M' = M[t\rangle$. (Hence $M[t\rangle$ denotes a marking after firing t at M and shows that t is enabled at M.) For IN, t is enabled at M if $M(p) \ge \alpha(p,t)$ for any $p \in {}^{\bullet}t$ and $M(q) = 0$ for any rivet q connected to t by an inhibitor arc. Let $\delta = t_{i_1} \cdots t_{i_s}$ be a sequence of transitions, and $\overline{\delta}(t)$ be the total number of occurrences of t in δ, where $T = \{t_1, \ldots, t_n\}$ and $i_j \in \{1, \ldots, n\}$. $\overline{\delta} = [\overline{\delta}(t_1) \cdots \overline{\delta}(t_n)]^{tr}$ $(n = |T|)$ is called the *firing count vector* of δ. Let $|\overline{\delta}|$ denote the sum of $\overline{\delta}(t)$ over all $t \in T$. For a marking M and an n-dimensional vector $X = [X(t_1) \cdots X(t_n)]^{tr}$, δ is called a *firing sequence* that is *legal* on M (denoted as $M[\delta\rangle$) if and only if t_{i_j} is enabled at M_{j-1} for $j = 1, \cdots, s$, where $M_0 = M$ and $M_j = M_{j-1}[t_{i_j}\rangle$. The resulting marking M_s also denotes $M[\delta\rangle$ for simplicity. Furthermore, for the markings M and M_s, and the firing sequence δ, $\langle\delta]M_s$ represents M. If $\overline{\delta} \le X$ for such δ then we say that δ is legal on M within X. A transition t is saturated (or unsaturated) in δ if $\overline{\delta}(t) = X(t)$ (or $\overline{\delta}(t) < X(t)$). Let $\delta\delta'$ denote concatenating δ' at the rear of δ for two firing sequences δ and δ'.

A directed cycle consisting of a pair of edges (p,t) and (t,p) is called a self-loop. In this paper, we assume that no self-loop exists in N (and in IN). N is called a conflict-free Petri net if and only if (i) or (ii) holds for any $p \in P$: (i) $|p^{\bullet}| \le 1$; (ii) any $t \in p^{\bullet}$ and p forms a self-loop. Since we assume that N has no self-loop, we consider only (i) for conflict-free Petri nets (which such a net is called a forward conflict-free Petri net). N is a *marked graph* if and only if any $p \in P$ has $|{}^{\bullet}p| \le 1$ and $|p^{\bullet}| \le 1$. Any marked graph is conflict-free.

III. An algorithm for MAX-INLFS

We show an algorithm *solve_INLFS_for_fcf* to solve **MAX-INLFS** when IN has weighted forward conflict-free structure (WFCF for short) structure.

An outline of the algorithm is as follows. Since $rv \in P'$ has some tokens, firing of any transition $t \in rv^{\circ}$ is prohibited and we consider **MAX-LFS** for N and X_v, where $X_v(t) \leftarrow 0$ for any $t \in P'^{\circ} \cup T_s$ and $X_v(t') \leftarrow X(t') - \overline{\delta}(t')$ for any $t' \in T - (P'^{\circ} \cup T_s)$. Then some rivets $rv \in P'$ may have no tokens. If such rivets exist then P' is updated and then we consier **MAX-LFS** as mentioned above again. This above operation is repeated as many as possible. Then one transition t_s, which is enabled, in T_s is selected and it fires. The above two operations are repeated as many as possible.

Now the description of the algorithm is given.

Algorithm *solve_INLFS_for_fcf*;
Input: An inhibitor-arc Petri net IN, an initial marking M_0, and a firing count vector X;
Output: A maximum firing sequence δ_m that is legal on M_0 within X;

1. $\delta_m \leftarrow$ (an empty sequence); $\delta \leftarrow$ (an empty sequence); $M \leftarrow M_0$;
2. *extend_sequence*(δ);

Procedure *extend_sequence*(δ);

1. $\delta_1 \leftarrow$ (an empty sequence); $\delta_2 \leftarrow$ (an empty sequence);
2. **while** $P' = \{rv \in RV \mid M(rv) > 0\} \neq \emptyset$ **do**
 2.1. Find a firing sequence δ_2 obtained by repeating firing of unsaturated enabled transitions $t \in T - (P'^{\circ} \cup T_s)$ beginning with a marking M as many times as possible, where $\overline{\delta\delta_1\delta_2}(t) \le X(t)$ for any $t \in T - (P'^{\circ} \cup T_s)$;
 2.2. $\delta_1 \leftarrow \delta_1\delta_2$; $M \leftarrow M[\delta_2\rangle$; /* Since each $t_t \in P'^{\bullet}$ fires as many times as possible, the number of tokens in $rv \in P'$ becomes as small as possible. */
 2.3. If there exist rivets $rv \in P$ having no token for the current marking M is P' then break this loop; otherwise, update P'; /* this loop is repeated */
3. $T'_s = \{t \in T_s \mid \overline{\delta\delta_1}(t) < X(t), t \text{ is enabled}\}$;
4. **while** $T'_s \neq \emptyset$ **do**
 4.1. Select t_s from T'_s; $T'_s \leftarrow T'_s \setminus \{t_s\}$;
 4.2. $M \leftarrow M[t_s\rangle$; /* fire t_s once */
 4.3. *extend_sequence*(δt_s);
 4.4. $M \leftarrow \langle t_s]M$; /* the resulting marking is $M_0[\delta\delta_1\rangle$ */
5. If every $t \in T$ satisfies $(\overline{\delta\delta_1}(t) = X(t))$ or $(\overline{\delta\delta_1}(t) < X(t)$ and t is not enabled at M) and $|\overline{\delta_m}| < |\overline{\delta\delta_1}|$ then $\delta_m \leftarrow \delta\delta_1$; /* If Step 4 executes then Step 5 does not execute */
6. $M \leftarrow \langle\delta_1]M$; /* the resulting marking is $M_0[\delta\rangle$ */ □

We will prove the next theorem.

Theorem 3.1: **MAX-INLFS** can be solved in $O(2^{|RV|}|P||X|)$ time if IN has WFCF structure. □

References

[1] J. L. Peterson: "Petri Net Theory and the Modeling of Systems", Prentice-Hall, Englewood Cliffs, N.J. (1981).

[2] T. Murata: "Petri nets: Properties, analysis and applications", Proc. IEEE, **77**, 4, pp. 541–580 (1989).

[3] S. Taoka and T. Watanabe: "Time complexity analysis of the legal firing sequence problem of petri nets with inhibitor arc", IEICE Trans. Fundamentals, **E89-A**, 11, pp. 3216–3226 (2006).

[4] S. Ochiiwa, S. Taoka and T. Watanabe: "Pseudo-polynomial time solvability for the maximum legal firing sequence problem of inhibitor-arc Petri nets –weighted marked graphs with inhibitor arcs–", Proc. ITC-CSCC 2012 (2012).

[5] S. Taoka, S. Ochiiwa and T. Watanabe: "Solvability for the maximum legal firing sequence problem of conflict-free petri nets with inhibitor arcs", Proc. ITC-CSCC 2014, pp. 861–864 (2014).

[6] M. R. Garey and D. S. Johnson: "Computers and Intractability: a Guide to the Theory of NP-Completeness", Freeman, San Francisco (1979).